A Simplified Algorithm for Finding Partially Invariant Solutions of Quasilinear Systems

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The method of partially invariant solutions of PDE systems was introduced by Ovsiannikov as a generalization of the classical similarity analysis. It offers a possibility to calculate exact solutions possessing a higher degree of freedom than similarity solutions. Ovsiannikov's algorithm, however, is somewhat hard to apply because one has to deal with three equation systems derived from the original PDE system. By means of the two-dimensional Euler equations, we show how the algorithm can be essentially simplified if classical similarity solutions are already known. Further, we prove a necessary criterion for the simplified algorithm to be senseful.

Key words: Hydrodynamics, Nonlinear systems, Group theory.

Similarity analysis is a well-known and useful method for finding exact group-invariant solutions of PDE systems [1, 2]. As a generalization, Ovsiannikov has introduced the theory of partially invariant solutions which have a higher degree of freedom than classical similarity solutions [1]. The algorithm described by Ovsiannikov yields, in general, two reduced PDE systems. It can well be applied to quasilinear first-order systems, e.g. the ideal MHD equations [3]. Martina and Winternitz calculated partially invariant solutions of nonlinear Klein-Gordon and Laplace equations [4]. But compared to the great number of publications on classical similarity solutions there exist very few examples of partially invariant solutions which are not actually invariant ones (we will call such solutions *proper partially invariant*) in the literature. This may be a result of the method's complexity.

In this paper, we develop a simplified algorithm for finding partially invariant solutions, which can be applied whenever classical similarity solutions are known. As a detailed similarity analysis has been done for many PDE systems of mathematical physics; this algorithm might offer a chance for finding a lot of further exact solutions. The algorithm, however, is not always senseful. Often it yields only solutions which do not differ essentially from the known similarity solutions. In order to avoid useless calculations we prove for the case of quasilinear first-order systems a necessary criterion for the algorithm to yield new, that means proper partially invariant solutions. We illustrate the algorithm by means of the Euler equations of two-dimensional incompressible hydrodynamics.

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1. Ovsiannikov's algorithm

We consider a first-order PDE system

$$F^i(x, u, u_x) = 0, \quad i = 1, \dots, l, \tag{1}$$

where $x = (x^1, \dots, x^n)$ denotes the independent variables, $u = (u^1, \dots, u^m)$ the dependent variables and $u_x = (u^1_{x^1}, \dots, u^m_{x^n})$ the $n\,m$-tupel of derivatives. Let G be an r-dimensional projectable symmetry group of (1) with the infinitesimal generators

$$X_i = \sum_{k=1}^{n} \alpha_i^k(x)\frac{\partial}{\partial x^k} + \sum_{j=1}^{m} \beta_i^j(x, u)\frac{\partial}{\partial u^j}; \quad i = 1, \dots, r.$$

The number $r_* := \max \operatorname{rank}(\alpha_i^k, \beta_i^j)$ is called the *geometrical dimension* of G. Let a solution of (1) be given by $u = \varphi(x)$. This m-dimensional equation determines an n-dimensional manifold in the $(n+m)$-dimensional space of variables, the solution's *graph* $\mathcal{S}_\varphi$. Its G-Orbit $\mathcal{O}_G(\mathcal{S}_\varphi)$ consists of all points of the space of variables, which are obtained from the graph's points by the transformations out of G:

$$\mathcal{O}_G(\mathcal{S}_\varphi) = \{(x, u) \in \mathbb{R}^{n+m} \mid (x, u) = g(\bar{x}, \bar{u}), \bar{u} = \varphi(\bar{x}), g \in G\}.$$

The difference between the dimensions of the graph's orbit and the graph itself is called the *defect* (of invariance) and denoted by δ: $\delta = \dim \mathcal{O}_G(\mathcal{S}_\varphi) - \dim \mathcal{S}_\varphi$. If φ is a G-invariant solution we have $\delta = 0$; in general the

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inequality $0 \leqq \delta \leqq \min\{r_*, m\}$ holds. If the defect is greater than zero but smaller than the maximal value, the solution is called *partially G-invariant*. The number $\varrho := n - r_* + \delta$ is called the *rank* (of invariance). If $G' \subset G$ is a subgroup and δ' and ϱ' denote the defect and rank with respect to G', it can be shown that $\varrho' \geqq \varrho$ and $\delta' \leqq \delta$. If a subgroup G' can be found such that $\varrho' = \varrho$ and $\delta' < \delta$, the solution is said to be *reducible*. If $\delta' = 0$ is possible, the solution is invariant under some subgroup of G. We denote partially invariant solutions which are not reducible to invariant ones as *proper partially invariant*. For finding partially G-invariant solutions with defect δ, Ovsiannikov gives the following algorithm (for details see [1]): First, the invariants of G are determined and introduced in the system as similarity variables. Contrary to the classical similarity analysis method, the "new" variables will not suffice to substitute all dependent variables (this is a result of the higher group dimension). A δ-tupel of "old" dependent variables must be preserved; these are called the *parametric variables*. The resulting equations form a system which is, in general, over-determined and can only be solved if some additional conditions which can be derived from it, are fulfilled. Because of this, the system is called the *active system*, the solvability conditions theirselves form the *reduced system*. It can be shown that this is always a PDE system for the similarity variables alone. That means that its solutions determine invariant manifolds in the space of variables. These are the possible *orbits* of partially invariant solutions. To locate the solutions inside the orbits, one has to substitute the orbit equations found by solving the reduced system into the original system (1). This yields a PDE system containing only the parametric variables and their derivatives. This remaining system has to be solved by conventional methods (similarity methods would only yield similarity solutions). In the case of a low defect its general solution can often be found. From this one gets the partially invariant solutions of the original system by re-substitution. In the case of quasilinear first-order systems, the described algorithm can well be formalized and applied by computers as described in [3].

2. Partially Invariant Solutions in the Orbit of a Known Similarity Solution

Ovsiannikov's algorithm in the described form can be applied to any PDE system if its maximal symmetry group (or a subgroup) is known. In this paper, however, we follow another idea: A given solution orbit, described by some solution of the reduced system will in general contain more than one partially invariant solution of the PDE system. Normally there will exist at least those solutions which are reducible to subgroup-invariant solutions. The idea is now the conversion of this fact: If some G'-invariant solution is known, its orbit under some symmetry group $G \supset G'$ can easily be calculated. In this orbit there may exist further PDE solutions, and these are all partially invariant with respect to G; some of them may be proper partially invariant ones. So, if a great deal of group-invariant solutions is known, the hard orbit calculation by setting up and solving the reduced system, as described in [3], becomes completely unnecessary.

Suppose that G is a symmetry group of (1) with the geometrical dimension r_* and $G' \subset G$ a subgroup with the dimension r'_*. Let $u = \varphi(x)$ be a G'-invariant solution and $\delta = r_* - r'_*$ its defect under G (in general, the inequality $\delta \leqq r_* - r'_*$ holds). Let $\omega^1, \ldots, \omega^t$, $t = n + m - r_*$, be the invariants of G. The orbit of some point $(\bar{x}, \bar{u})$ is defined by the constance of the invariants, i.e. by the equations $\omega^j(x,u) = \omega^j(\bar{x}, \bar{u})$, $j = 1, \ldots, t$. So the solution orbit can easily be calculated by eliminating $\bar{x}$ from the equations $\omega^j(x,u) = \omega^j(\bar{x}, \varphi(\bar{x}))$. As the orbit's dimension is $n + \delta$, the obtained equations are of the form

$$\psi^i(x,u) = 0, \quad i = 1, \ldots, m - \delta =: \mu, \quad \operatorname{rank} \frac{\partial \psi^i}{\partial (x,u)} = \mu. \tag{2}$$

It can be shown that even the harder rank condition $\operatorname{rank} \frac{\partial \psi^i}{\partial u} = \mu$ always holds. Hence, the orbit equations (2) can be solved for μ dependent variables, say

$$u^i = f^i(x, u^{\mu+1}, \ldots, u^m), \quad i = 1, \ldots, \mu.$$

From this we get

$$u^i_{x^j} = \frac{\partial f^i}{\partial x^j} + \sum_{k=\mu+1}^{m} \frac{\partial f^i}{\partial u^k} u^k_{x^j}$$

and higher derivatives analogously. By substituting these results into the original system (1) one gets some differential equations for the parametric variables $u^p := (u^{\mu+1}, \ldots, u^m)$, determining just those solutions which lie in the orbit of the given solution φ. This system is identical to the remaining system, which would have been yielded by the algorithm described in [3].

3. A Necessary Criterion for the Existence of Proper Partially Invariant Solutions

In many practical cases the described procedure yields only similarity solutions which do not differ essentially from the given solution φ. In order to avoid needless calculations it seems desirable to possess a criterion which allows the early recognition of such cases.

If proper partially invariant solutions are found, they *must* be new, even if a complete similarity analysis of the system has been done before.

In the following, we prove a *necessary* criterion for the existence of proper partially invariant solutions in the orbit of a given similarity solution in the case of quasilinear first-order PDE systems. For this purpose we first have to provide Ovsiannikov's reduction theorem.

Let $u = \varphi(x)$ be a partially G-invariant solution of (1) and δ its defect. The solution's orbit is then described by $m - \delta$ independent equations $w^\nu(x, u) = 0$, $\nu = 1, \ldots, m - \delta$ in the space of variables. Therefore, the relations

$$\frac{\partial w^\nu}{\partial x^j} + \sum_{i=1}^{m} \frac{\partial w^\nu}{\partial u^i} u^i_{x^j} = 0, \quad \nu = 1, \ldots, m-1,\ j = 1, \ldots, n \tag{3}$$

are fulfilled on the solution.

Theorem (Ovsiannikov): If the PDE system (1) together with the equations (3) can be uniquely solved for all first derivatives in the form

$$u^i_{x^j} = f^i_j(x, u), \quad i = 1, \ldots, m, \quad j = 1, \ldots, n,$$

then there exists a subgroup $G' \subset G$ such that φ is a G'-invariant solution. (For the proof see [1].)

Now we are able to prove the following

Theorem: Let a PDE system be given by

$$\sum_{i=1}^{m} \sum_{j=1}^{n} A^j_{ki}(x, u)\, u^i_{x^j} = B_k(x, u), \quad k = 1, \ldots, l. \tag{4}$$

Let G' and $G \supset G'$ be intransitive symmetry groups of (4) with the generators $X_1, \ldots, X_r$ resp. $X_1, \ldots, X_r, Y$. Let $u = \varphi(x)$ be a G'-invariant but not G-invariant solution, i.e. $\delta = 1$. Then the inequality

$$\operatorname{rank}\left(\sum_{i=1}^{m} A^j_{ki}(x, \varphi(x))\, Y(\varphi^i(x) - u^i)|_{u=\varphi(x)} \right)_{\substack{j=1,\ldots,n \\ k=1,\ldots,l}} < n$$

is necessary for the existence of proper partially invariant solutions in the orbit of φ, so far as the linear equation system given by (3) and (4) has a unique rank on the whole orbit (this requirement is hard to check in praxis, but not very important and often verifiable subsequently).

Proof. The orbit of the given solution φ is described by

$$w^\nu(x, u) = 0, \quad \nu = 1, \ldots, m-1,$$
$$\operatorname{rank} \frac{\partial(w^1, \ldots, w^{m-1})}{\partial(u^1, \ldots, u^m)} = m - 1. \tag{5}$$

We can assume

$$\det \frac{\partial(w^1, \ldots, w^\nu)}{\partial(u^1, \ldots, u^{m-1})} \neq 0.$$

Hence, there exists the inverse matrix W, i.e.

$$\sum_{\nu=1}^{m-1} \frac{\partial w^\nu}{\partial u^i} W^{i'}_\nu(x, u) = \delta^{i'}_i, \quad i', i = 1, \ldots, m-1.$$

In the orbit we have

$$\frac{\partial w^\nu}{\partial x^j} + \sum_{i=1}^{m} \frac{\partial w^\nu}{\partial u^i} u^i_{x^j} = 0 \tag{6}$$

$$\Rightarrow \sum_{\nu=1}^{m-1} \frac{\partial w^\nu}{\partial x^j} W^{i'}_\nu + \underbrace{\sum_{i=1}^{m-1} \sum_{\nu=1}^{m-1} \frac{\partial w^\nu}{\partial u^i} W^{i'}_\nu}_{=\delta^{i'}_i} u^i_{x^j} + \sum_{\nu=1}^{m-1} \frac{\partial w^\nu}{\partial u^m} u^m_{x^j} = 0$$

$$\Rightarrow u^i_{x^j} = -\sum_{\nu=1}^{m-1} \left(\frac{\partial w^\nu}{\partial x^j} + \frac{\partial w^\nu}{\partial u^m} u^m_{x^j} \right) W^i_\nu, \quad i = 1, \ldots, m-1, \quad j = 1, \ldots, n. \tag{7}$$

By substituting this into the PDE system (4) we get

$$-\sum_{j=1}^{n} \sum_{i=1}^{m} A^j_{ki} \sum_{\nu=1}^{m-1} \left(\frac{\partial w^\nu}{\partial x^j} + \frac{\partial w^\nu}{\partial u^m} u^m_{x^j} \right) W^i_\nu + \sum_{j=1}^{n} A^j_{ki} u^m_{x^j} = B_k$$

$$\Rightarrow K^j_k u^m_{x^j} = B_k + \sum_{j=1}^{n} \sum_{i=1}^{m-1} \sum_{\nu=1}^{m-1} A^j_{ki} \frac{\partial w^\nu}{\partial x^j} W^i_\nu \tag{8}$$

with

$$K^j_k \equiv K^j_k(x, u) = A^j_{ki}(x, u) - \sum_{i=1}^{m-1} \sum_{\nu=1}^{m-1} A^j_{ki}(x, u) \frac{\partial w^\nu}{\partial u^m} W^i_\nu(x, u). \tag{9}$$

If (8) is solvable for all $u^m_{x^j}$, $j = 1, \ldots, n$, all first derivatives are determined by (7) and the reduction theorem excludes the existence of proper partially invariant solutions. This case is given if the rank condition

$$\operatorname{rank}(K_k^j(x,u))_{\substack{j=1,\dots,n\\ k=1,\dots,l}}=n \tag{10}$$

holds. As this rank is required to be constant on the whole orbit, it is sufficient to calculate it on the given solution $u=\varphi(x)$. As

$$Y=\sum_{j=1}^{n}\alpha^j(x,u)\frac{\partial}{\partial x^j}+\sum_{i=1}^{m}\beta^i(x,u)\frac{\partial}{\partial u^i}$$

is an infinitesimal generator of G and (5) defines a G-invariant manifold containing the graph of φ, the equation

$$Y w^\nu(x,\varphi(x))=\left.\left(\sum_{j=1}^{n}\alpha^j\frac{\partial w^\nu}{\partial x^j}+\sum_{i=1}^{m}\beta^i\frac{\partial w^\nu}{\partial u^i}\right)\right|_{u=\varphi(x)}=0$$

holds. On the other hand we have

$$\left.\left(\sum_{j=1}^{n}\alpha^j\frac{\partial w^\nu}{\partial x^j}+\sum_{j=1}^{n}\sum_{i=1}^{m}\alpha^j\frac{\partial w^\nu}{\partial u^i}\frac{\partial\varphi^i}{\partial x^j}\right)\right|_{u=\varphi(x)}=0$$

because of (6). From this we get

$$\left.\left(\sum_{i=1}^{m}\left(\sum_{j=1}^{n}\alpha^j\frac{\partial\varphi^i}{\partial x^j}-\beta^i\right)\frac{\partial w^\nu}{\partial u^i}\right)\right|_{u=\varphi(x)}=0,\ \nu=1,\dots,m-1.$$

With the abbreviation

$$\lambda^i(x):=\sum_{j=1}^{n}\alpha^j(x,\varphi(x))\frac{\partial\varphi^i}{\partial x^j}-\beta^i(x,\varphi(x))$$
$$=Y(\varphi^i(x)-u^i)|_{u=\varphi(x)}$$

that means

$$\sum_{i=1}^{m}\lambda^i(x)\left.\frac{\partial w^\nu}{\partial u^i}\right|_{u=\varphi(x)}=0$$

$$\Rightarrow\sum_{\nu=1}^{m-1}\lambda^i(x)\underbrace{\sum_{\nu'=1}^{m-1}\left.\frac{\partial w^{\nu'}}{\partial u^i}\right|_{u=\varphi(x)}W_{\nu'}^{i'}(x,\varphi(x))}_{=\delta_i^{i'}}$$

$$+\sum_{\nu=1}^{m-1}\lambda^m(x)\left.\frac{\partial w^\nu}{\partial u^m}\right|_{u=\varphi(x)}W_\nu^{i'}(x,\varphi(x))=0$$

$$\Rightarrow\lambda^i(x)=-\lambda^m(x)\sum_{\nu=1}^{m-1}\left.\frac{\partial w^\nu}{\partial u^m}\right|_{u=\varphi(x)}W_\nu^i(x,\varphi(x)),$$
$$i=1,\dots,m-1.$$

As $\lambda^m(x)\neq 0$ (otherwise $\lambda^i(x)=0$ would follow for $i=1,\dots,m$ and the solution would be G-invariant in contradiction to the requirements) we can multiply (9) by this factor without losing information.

Substituting $u=\varphi(x)$ we get

$$\lambda^m(x)K_k^j(x,\varphi(x))$$
$$=A_{km}^j(x,\varphi(x))\lambda^m(x)$$
$$-\sum_{i=1}^{m-1}A_{ki}^j(x,\varphi(x))\underbrace{\lambda^m(x)\sum_{\nu=1}^{m-1}\left.\frac{\partial w^\nu}{\partial u^m}\right|_{u=\varphi(x)}W_\nu^i(x,\varphi(x))}_{=-\lambda^i(x)}$$
$$=\sum_{i=1}^{m}A_{ki}^j(x,\varphi(x))\lambda^i(x)$$
$$=\sum_{i=1}^{m}A_{ki}^j(x,\varphi(x))\,Y(\varphi^i(x)-u^i)|_{u=\varphi(x)}.$$

Therefore, the rank of the coefficient matrix (K_k^i) is equal to the rank of the right side. So the theorem statement follows by negation of (10). □

The proven criterion is of purely differential character, that means that it can be applied directly after establishing the Lie algebras of G and G' and the solution φ without doing any integrations.

4. Example

We consider the two-dimensional Euler equations of ideal incompressible hydrodynamics:

$$\begin{aligned} &u_t+u u_x+v u_y+p_x=0,\\ &v_t+u v_x+v v_y+p_y=0,\\ &u_x+v_y=0,\end{aligned}\tag{11}$$

where u and v denote the velocity components in x- resp. y-direction, t the time and p the pressure. A classical similarity analysis of the corresponding three-dimensional system has been done by Olver [2]. For our purpose it will be sufficient to give two symmetry generators:

$$X_g=g(t)\frac{\partial}{\partial x}+g'(t)\frac{\partial}{\partial u}-g''(t)\,x\frac{\partial}{\partial p},$$

$$X_h=h(t)\frac{\partial}{\partial y}+h'(t)\frac{\partial}{\partial v}-h''(t)\,y\frac{\partial}{\partial p}$$

with twice differentiable time functions g and h. As their commutator vanishes, these two generators span the Lie algebra of a two-parameter symmetry group G which contains the one-parameter group G' with the generator X_g as a subgroup. The general G'-invariant solution is

$$u=\frac{f(\chi(t)-y\,g(t))+g'(t)}{g(t)}=:\varphi^1(x,y,t),$$

$$v = \frac{\chi'(t) - y\,g'(t)}{g(t)} =: \varphi^2(x,y,t),$$

$$p = \frac{y^2}{2}\frac{g(t)g''(t) - 2g'^2(t)}{g^2(t)} + y\frac{2\chi'(t)g'(t) - g(t)\chi''(t)}{g^2(t)}$$

$$-\frac{g''(t)}{g(t)}\frac{x^2}{2} + \psi(t) =: \varphi^3(x,y,t) \qquad (12)$$

with two other free time functions and a free function f. We now want to find further solutions in the G-orbit of the given solution. First, we use our criterion to determine the free functions in such a way that proper partially invariant solutions can be found principally. The matrices A_k, $k = 1,2,3$, can easily be read from (11):

$$A_1 = \begin{pmatrix} u & 0 & 1 \\ v & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \; A_2 = \begin{pmatrix} 0 & u & 0 \\ 0 & v & 1 \\ 0 & 1 & 0 \end{pmatrix}, \; A_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

With $Y = X_h$ we get

$$Y(\varphi^1(x,y,t) - u) = -h(t)f'(\chi(t) - y\,g(t)),$$

$$Y(\varphi^2(x,y,t) - v) = -h(t)\frac{g'(t)}{g(t)} - h'(t),$$

$$Y(\varphi^3(x,y,t) - p) = y\,h(t)\frac{g(t)g''(t) - 2g'^2(t)}{g^2(t)} + h(t)\frac{2\chi'(t)g'(t) - g(t)\chi''(t)}{g^2(t)} + y\,h''(t).$$

Therefore, the criterion is fulfilled if

$$\begin{vmatrix} -uhf' + yh\frac{gg''-2g'^2}{g^2} + h\frac{2\chi'g'-g\chi''}{g^2} + yh'' & -uh\frac{g'}{g} - uh' & -hf' \\ -vhf' & -vh\frac{g'}{g} - vh' + yh\frac{gg''-2g'^2}{g^2} + h\frac{2\chi'g'-g\chi''}{g^2} + yh'' & -h\frac{g'}{g} - h' \\ -hf' & -h\frac{g'}{g} - h' & 0 \end{vmatrix}$$

$$= \left((hf')^2 + \left(h\frac{g'}{g} + h'\right)^2\right)\cdot\left(yh\frac{gg''-2g'^2}{g^2} + h\frac{2\chi'g'-g\chi''}{g^2} + yh''\right) = 0. \qquad (13)$$

So there are two possibilities to satisfy the criterion. We first assume that the second factor vanishes. As $h = 0$ is senseless, that means

$$\frac{g(t)g''(t) - 2g'^2(t)}{g^2(t)} + \frac{h''(t)}{h(t)} = 0$$

or

$$g(t)\frac{d^2}{dt^2}\frac{1}{g(t)} = \frac{h''(t)}{h(t)} \qquad (14\,a)$$

and

$$2\chi'(t)g'(t) - g\chi''(t) = 0. \qquad (14\,b)$$

The first differential equation is solved by $g(t)\cdot h(t) =$ const, but we want to exclude this possibility in the following (a solution of another type is e.g. $h(t) = \sin t$, $g(t) = \frac{1}{\cos t}$). The invariants $\omega^1, \ldots, \omega^4$ of G are found as

$$\omega^1 = t, \quad \omega^2 = u - \frac{g'(t)}{g(t)}x, \quad \omega^3 = v - \frac{h'(t)}{h(t)}y,$$

$$\omega^4 = \frac{h''(t)}{h(t)}\frac{y^2}{2} + \frac{g''(t)}{g(t)}\frac{x^2}{2}. \qquad (15)$$

To determine the G-orbit of the solution (12), we calculate the invariants on this solution, having regard to (14):

$$\omega^1 = t, \quad \omega^2 = \frac{f(\chi(t) - y\,g(t))}{g(t)},$$

$$\omega^3 = \frac{\chi'(t)}{g(t)} - \left(\frac{g'(t)}{g(t)} + \frac{h'(t)}{h(t)}\right)y, \quad \omega^4 = \psi(t). \qquad (16)$$

The statement $g\cdot h \neq$ const is equivalent to $\frac{g'}{g} + \frac{h'}{h} \neq 0$. The orbit must be representable by $m - \delta = 2$ equations containing only invariants. These equations are found by eliminating t and y from (16) as

$$\omega^2 = \frac{1}{g(t)}\cdot f\left(\chi(\omega^1) - \frac{\chi'(\omega^1) - \omega^3 g(\omega^1)}{\frac{g'(\omega^1)}{g(\omega^1)} + \frac{h'(\omega^1)}{h(\omega^1)}}\right),$$

$$\omega^4 = \psi(\omega^1).$$

We re-substitute (15) and solve the resulting equations for u and p, keeping v as parametric variable:

$$u = \frac{f(B(t,y,v)) + x\,g'(t)}{g(t)}, \quad v = v,$$

$$p = \psi(t) - \frac{h''(t)}{h(t)}\frac{y^2}{2} - \frac{g''(t)}{g(t)}\frac{x^2}{2} \tag{17}$$

with the abbreviation

$$B(t,y,v) := \chi(t) - \frac{\chi'(t) - g(t)\left(v - \dfrac{h'(t)}{h(t)}y\right)}{\dfrac{g'(t)}{g(t)} + \dfrac{h'(t)}{h(t)}}.$$

Substituting our results in the Euler equations (11) yields a quasilinear PDE system for v:

$$f'(B)\frac{\partial B}{\partial v}\left(v_t + \frac{f(B) + x\,g'(t)}{g(t)}v_x + v\,v_y\right) = -f'(B)\left(\frac{\partial B}{\partial t} + v\frac{\partial B}{\partial y}\right),$$

$$v_t + \frac{f(B) + x\,g'(t)}{g(t)}v_x + v\,v_y = \frac{h''(t)}{h(t)}y,$$

$$\frac{f'(B)}{g(t)}\frac{\partial B}{\partial v}v_x + v_y = -\frac{g'(t)}{g(t)}. \tag{18}$$

The coefficient matrix for (v_x, v_y, v_t) actually has the rank 2 on the whole space of variables. It can be shown, that with (14) the first equation is a multiple of the second. Instead of trying to find the general solution of the system (18), we simplify it by setting $f = C = \text{const}$. This yields with (14) the equations

$$v_t + \frac{C + g'(t)\,x}{g(t)}v_x + v\,v_y = -\frac{g(t)g''(t) - 2g'^2(t)}{g^2(t)}y,$$

$$v_y = -\frac{g'(t)}{g(t)},$$

The second one is easy to integrate:

$$v = -\frac{g'(t)}{g(t)}y + j(x,t).$$

By substituting this in the first equation we get a PDE for the function j:

$$j_t + \frac{C + x\,g'(t)}{g(t)}j_x - \frac{g'(t)}{g(t)}j = 0.$$

It can be integrated by the method of characteristics; its general solution is

$$j(x,t) = F\left(C\int\frac{dt}{g^2(t)} - \frac{x}{g(t)}\right)\cdot g(t),$$

where F denotes an arbitrary one-argument function. Regarding (17) we get a set of partially G-invariant solutions of (11):

$$v = F\left(C\int\frac{dt}{g^2(t)} - \frac{x}{g(t)}\right)\cdot g(t) - \frac{g'(t)}{g(t)}y,$$

$$u = \frac{C + x\,g'(t)}{g(t)},$$

$$p = \psi(t) + \frac{g(t)g'(t) - 2g'^2(t)}{g^2(t)}\frac{y^2}{2} - \frac{g''(t)}{g(t)}\frac{x^2}{2}.$$

These solutions are in fact proper partially invariant if F is non-constant. To verify this, we apply a linear combination of X_g and X_h to them:

$$[a\,X_g + b\,X_h]\left(F\left(C\int\frac{dt}{g^2(t)} - \frac{x}{g(t)}\right)\cdot g(t) - \frac{g'(t)}{g(t)}y - v\right)$$

$$= -a\,g(t)F'\left(C\int\frac{dt}{g^2(t)} - \frac{x}{g(t)}\right) - b\,h(t)\frac{g'(t)}{g(t)} - b\,h'(t).$$

As we required $g'/g + h'/h \neq 0$, this term can only vanish if $b = 0$ and $F = \text{const}$. Otherwise there is no subgroup of G leaving the solutions invariant.

The second possibility to satisfy (13) is to set the first factor to zero. This means $f = C = \text{const}$ and $g'/g + h'/h = 0$ or $g \cdot h = \text{const}$. We now want to consider this case which we had excluded so far. We can set $h(t) = 1/g(t)$ and the generator Y gets the form

$$Y = \frac{1}{g(t)}\frac{\partial}{\partial y} - \frac{g'(t)}{g^2(t)}\frac{\partial}{\partial v} + y\frac{g(t)g''(t) - 2g'^2(t)}{g^3(t)}\frac{\partial}{\partial p}.$$

The invariants of G are

$$\omega^1 = t, \quad \omega^2 = u - \frac{g'(t)}{g(t)}x, \quad \omega^3 = v + \frac{g'(t)}{g(t)}y,$$

$$\omega^4 = \frac{y^2}{2}\left(\frac{2g'^2(t)}{g(t)g''(t)} - 1\right) + \frac{x^2}{2} + \frac{g(t)}{g''(t)}p. \tag{19}$$

On the given solution (12) they are evaluated as

$$\omega^1 = t, \quad \omega^2 = \frac{C}{g(t)}, \quad \omega^3 = \frac{\chi'(t)}{g(t)},$$

$$\omega^4 = y\frac{2\chi'(t)g'(t) - g(t)\chi''(t)}{g(t)g''(t)} + \frac{g(t)}{g''(t)}\psi(t).$$

By eliminating t and y we get the equations describing the given solution's G-orbit

$$\omega^2 = \frac{C}{g(\omega^1)}, \quad \omega^3 = \frac{\chi'(\omega^1)}{g(\omega^1)}.$$

We substitute (19) and solve for u and v (p must play the role of the parametric variable as it does not appear in the equations):

$$u = \frac{C}{g(t)} + \frac{g'(t)}{g(t)} x, \quad v = \frac{\chi'(t)}{g(t)} - \frac{g'(t)}{g(t)} y. \tag{20}$$

Substituting (20) in (11) yields the PDE system for p:

$$\frac{g''(t)}{g^2(t)} x + p_x = 0,$$

$$- y \frac{g(t) g''(t) - 2 g'^2(t)}{g^2(t)} + \frac{2 \chi'(t) g'(t) - \chi''(t) g(t)}{g^2(t)} + p_y = 0.$$

The coefficient matrix for (p_x, p_y, p_t) again has the rank 2 on the space of variables. Nevertheless, it is clear that solving the system will only reproduce the given solution (12) because the parametric variable has no influence on u and v. This example shows that our criterion is in fact only necessary but not sufficient for the existence of proper partially invariant solutions.

[1] L. V. Ovsiannikov, Group Analysis of Differential Equations, Academic Press, New York 1982.
[2] P. J. Olver, Applications of Lie groups to Differential Equations, Springer, New York 1986.
[3] D.-A. Becker and E. W. Richter, Z. Naturforsch. **45a,** 1219 (1990).
[4] L. Martina and P. Winternitz, J. Math. Phys. **33,** 2718 (1992).